International Journal of Advanced Research in Engineering Technology & Science

ISSN: 2349-2819

Email: editor@ijarets.org Volume-9, Issue-6 June – 2022 www.ijarets.org

NUMERICAL INVESTIGATION OF FRACTIONAL-ORDER PARTIAL DIFFERENTIAL EQUATIONS VIA GENERALIZED NARAYANA POLYNOMIALS

Dr. Ajay Kumar Mishra
Director Research & Development
Asian International University
Ghari, Imphal, Manipur

ABSTRACT

Fractional-order partial differential equations (FPDEs) are important in the modeling of complex phenomena with memory and hereditary effects in applications in the fields of anomalous diffusion, viscoelastic materials, heat transfer, fluid flow, biological systems, and financial modeling. As a result of a nonlocal character of fractional derivatives, analytical solutions of FPDEs are seldom known, thus requiring the invention of effective and precise numerical methods. In the current paper, it is suggested to use a spectral collocation-based numerical scheme based on generalized Narayana polynomials, to solve a group of FPDEs written in the Caputo form. It is estimated by a finite series expansion of generalized Narayana polynomials and matrices of operational forms of fractional and integer-order derivatives are built to convert the equations of interest into systems of algebraic equations. The convergence and error analysis of theoretical convergence are developed so that the approach is reliable. To illustrate the accuracy, the stability and the computational efficiency of the proposed scheme several numerical experiments of linear and nonlinear partial differential equations of the same order but of varying degrees of fractionality are reported. The numerical outcomes demonstrate the fast convergence of the error and great consistency with the analytical solutions, which proves the spectral-rate convergence and strength of the generalized Narayana polynomials methodology.

Keywords: *Fractional-order partial differential equations, Generalized Narayana polynomials, Spectral collocation method, Operational matrix, Numerical convergence.*

1. INTRODUCTION

The extension of classical differentiation and integration to non-integer orders through fractional calculus has received much interest because it can model memory and hereditary behavior of complex systems. Fractional-order partial differential equations occur naturally in anomalous diffusion, viscoelastic materials, heat transfer, fluid flow, biological systems and financial modeling. In contrast to integer-order models, the fractional models are better fitting experimental and observed data.

Frankly speaking, however, FPDEs are extremely challenging to solve analytically because of the nonlocal qualities of the fractional derivatives. This has resulted in the establishment of strong and effective numerical techniques as a primary research topic. The conventional numerical methods of finite difference and finite element methods tend to demand a dense grid of discretization leading to high computational cost.

Polynomial-based spectral and collocation methods have been developed as an alternative and have exponential convergence of smooth solutions. Fractional differential equations have more recently been solved using generalized polynomials (Bernoulli, Laguerre, Fibonacci, MorganVoyce and Narayana polynomials).

Driven by these advances, a generalized Narayana poly-based numerical framework of FPDEs is proposed in this paper. The key contributions of this work are:

1. Building of a working matrix of fractional derivatives using generalized Narayana polynomials.
2. A spectral collocation method of solving FPDEs.
3. Close convergence and error analysis.
4. Numerical verification with benchmark cases and connection with existing procedures.

2. LITERATURE REVIEW

Dinh et al. (2019) investigated model reduction techniques for fractional elliptic problems using Kato's formula and developed a mathematically rigorous framework to simplify high-dimensional fractional systems. The study addressed the computational challenges associated with fractional operators, which typically involve nonlocal behavior and high computational cost. By applying model reduction strategies, the authors demonstrated that it was possible to construct reduced-order representations that preserved the essential dynamics and accuracy of the original system. The findings highlighted that the proposed approach significantly improved computational efficiency, making it suitable for large-scale simulations and practical applications in scientific computing, engineering, and applied mathematics.

Hassani et al. (2020) examined the solution of nonlinear systems of fractional-order partial differential equations using an optimization technique based on generalized polynomials and proposed an innovative computational methodology. The study combined polynomial approximation methods with optimization algorithms to effectively solve complex nonlinear fractional systems characterized by memory and nonlocal effects. It demonstrated that the approach improved convergence rates and reduced computational complexity compared to traditional methods. The results also showed that the technique was capable of accurately capturing the intricate behavior of fractional systems, thereby contributing to the advancement of numerical methods in fractional calculus and their applications in physics and engineering.

Nemati et al. (2021) explored numerical techniques for solving variable-order fractional differential equations using Bernoulli polynomials and developed an efficient and reliable algorithm for handling such problems. The study emphasized that variable-order fractional models provided a more realistic representation of physical phenomena, as they allowed the order of differentiation to change with time or space. By utilizing Bernoulli polynomials, the authors demonstrated improved numerical stability, faster convergence, and high accuracy in approximating solutions. The research highlighted the effectiveness of the method in dealing with complex dynamical systems involving variable memory effects, making it a valuable contribution to computational fractional calculus.

Rusagara (2015) conducted a detailed numerical investigation of heat transfer in one-dimensional longitudinal fins, focusing on the analysis of temperature distribution and heat dissipation under varying physical conditions. The study employed differential equation models along with numerical techniques to simulate heat transfer processes and evaluate the efficiency of fin designs. It demonstrated how parameters such as fin geometry, material properties, and boundary conditions influenced thermal performance. The findings emphasized the importance of accurate mathematical modeling and numerical analysis in optimizing engineering designs, particularly in thermal systems where efficient heat dissipation is critical.

3. MATHEMATICAL PRELIMINARIES

The following section provides the concepts and notations of the field of fractional calculus and generalized Narayana polynomials necessary to formulate the suggested numerical scheme.

3.1 Fractional Derivative

Let $f(t) \in C^1[0,T]$. The Caputo fractional derivative of order $\alpha \in (0,1)$is defined by

$$cD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau. \qquad (3.1)$$

Caputo formulation is especially beneficial when modeling physical processes, since it permits the use of classical initial conditions of integer order e.g.

$$f(0) = f_0, \qquad (3.2)$$

which is not generally possible under the Riemann–Liouville definition.

For a power function $f(t) = t^{\beta}$, where $\beta > -1$, the Caputo fractional derivative is given by

$$cD_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \beta \geq 1. \qquad (3.3)$$

Furthermore, the Caputo derivative satisfies the linearity property

$$\mathrm{c}D_t^{\alpha}[af(t) + bg(t)] = a\, \mathrm{c}D_t^{\alpha} f(t) + b\, \mathrm{c}D_t^{\alpha} g(t), \qquad (3.4)$$

where $a$and $b$are constants. This property plays a crucial role in constructing operational matrices for numerical approximations.

3.2 Generalized Narayana Polynomials

Definition 3.1

The generalized Narayana polynomials

$$\left\{N_n^{(\lambda)}(x)\right\}_{n=0}^{\infty}$$

are defined recursively as

$$N_0^{(\lambda)}(x) = 1, N_1^{(\lambda)}(x) = x, \qquad (3.5)$$

$$N_{n+1}^{(\lambda)}(x) = (x+\lambda)N_n^{(\lambda)}(x) - \lambda N_{n-1}^{(\lambda)}(x), n \geq 1. \qquad (3.6)$$

Using the recurrence relation, the first few generalized Narayana polynomials are obtained as

$$N_2^{(\lambda)}(x) = x^2 + \lambda x - \lambda, \qquad (3.7)$$
$$N_3^{(\lambda)}(x) = x^3 + 2\lambda x^2 - \lambda x - \lambda^2. \qquad (3.8)$$

Each polynomial $N_n^{(\lambda)}(x)$is of degree n, and the set $\left\{N_n^{(\lambda)}(x)\right\}$forms a linearly independent basis on bounded intervals.

A sufficiently smooth function $u(x)$defined on $[0, L]$can be approximated as

$$u(x) \approx \sum_{n=0}^{m} c_n N_n^{(\lambda)}(x), \qquad (3.9)$$

where $c_n$are unknown coefficients determined through collocation or projection techniques.

Remark 3.1

The generalized Narayana polynomials exhibit simple recurrence relations, polynomial growth, and numerical stability, making them particularly suitable for spectral and collocation-based numerical methods.

Remark 3.2

Due to their polynomial structure, the derivatives of generalized Narayana polynomials can be expressed in terms of the same basis, which facilitates the construction of operational matrices for integer and fractional derivatives

Remark 3.3

The completeness of the generalized Narayana polynomial basis ensures that the approximation error decreases rapidly as the truncation order increases, leading to spectral-rate convergence for smooth solutions.

4. APPROXIMATION AND OPERATIONAL MATRIX

Let $u(x,t)$ denote the solution of a fractional-order partial differential equation defined on the space–time domain

$$\Omega = [0,L] \times [0,T]. \qquad (4.1)$$

The solution $u(x,t)$ is approximated by a finite series expansion in terms of generalized Narayana polynomials as

$$u_m(x,t) = \sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij}\, N_i^{(\lambda)}(x)\, N_j^{(\lambda)}(t), \qquad (4.2)$$

where c_{ij} are unknown real coefficients to be determined.

4.1 Matrix Representation of the Approximation

Define the Narayana polynomial vectors

$$\mathbf{N}_x(x) = [N_0^{(\lambda)}(x), N_1^{(\lambda)}(x), \dots, N_m^{(\lambda)}(x)], \qquad (4.3)$$
$$\mathbf{N}_t(t) = [N_0^{(\lambda)}(t), N_1^{(\lambda)}(t), \dots, N_m^{(\lambda)}(t)]^T. \qquad (4.4)$$

Let

$$\mathbf{C} = \left[c_{ij}\right]_{(m+1)\times(m+1)} \qquad (4.5)$$

be the coefficient matrix. Then the approximate solution (4.2) can be written compactly as

$$u_m(x,t) = \mathbf{N}_x(x)\, \mathbf{C}\, \mathbf{N}_t(t). \qquad (4.6)$$

This matrix formulation significantly simplifies the implementation of the numerical scheme.

4.2 Operational Matrix of Fractional Derivative

To handle the fractional derivative with respect to time, an operational matrix corresponding to the Caputo derivative is constructed.

Theorem 4.1 (Operational Matrix of Fractional Derivative)

Let

$$\mathbf{N}_t(t) = [N_0^{(\lambda)}(t), N_1^{(\lambda)}(t), \dots, N_m^{(\lambda)}(t)]^T. \qquad (4.7)$$

Then, there exists an operational matrix

$$\mathbf{D}^{(\alpha)} \in \mathbb{R}^{(m+1)\times(m+1)} \qquad (4.8)$$

such that the Caputo fractional derivative of order $\alpha \in (0,1)$can be approximated by

$$cD_t^{\alpha}\mathbf{N}_t(t) \approx \mathbf{D}^{(\alpha)}\mathbf{N}_t(t). \qquad (4.9)$$

Proof

Using the linearity of the Caputo fractional derivative, the derivative of each basis function $N_j^{(\lambda)}(t)$can be written as

$$cD_t^{\alpha}N_j^{(\lambda)}(t) = \sum_{k=0}^{\infty} d_{jk}^{(\alpha)}\, N_k^{(\lambda)}(t), \qquad (4.10)$$

where $d_{jk}^{(\alpha)}$are expansion coefficients.

By truncating the infinite series to the first $m+1$terms and projecting onto the finite-dimensional Narayana polynomial space, we obtain

$$cD_t^{\alpha} N_j^{(\lambda)}(t) \approx \sum_{k=0}^{m} d_{jk}^{(\alpha)} N_k^{(\lambda)}(t), j = 0,1, \dots, m. \qquad (4.11)$$

Collecting the coefficients $d_{jk}^{(\alpha)}$yields the operational matrix

$$\mathbf{D}^{(\alpha)} = \left[d_{jk}^{(\alpha)}\right]_{(m+1)\times(m+1)}. \qquad (4.12)$$

Hence, equation (4.9) follows directly.

4.3 Fractional Derivative of the Approximate Solution

Applying the operational matrix to the approximate solution (4.6), the fractional derivative with respect to time is given by

$$cD_t^{\alpha} u_m(x,t) = \mathbf{N}_x(x)\, \mathbf{C}\, \mathbf{D}^{(\alpha)} \mathbf{N}_t(t). \qquad (4.13)$$

Similarly, integer-order spatial derivatives can be represented using corresponding operational matrices:

$$\frac{\partial^r u_m(x,t)}{\partial x^r} = \mathbf{N}_x(x)\, \mathbf{D}_x^{(r)}\, \mathbf{C}\, \mathbf{N}_t(t), r = 1,2, \dots \qquad (4.14)$$

Remark 4.1

The operational matrix approach transforms the original fractional-order partial differential equation into a system of algebraic equations in terms of the unknown coefficient matrix **C**, significantly reducing computational complexity.

Remark 4.2

The accuracy of the approximation improves rapidly as the truncation order $m$increases, and spectral-rate convergence is achieved for sufficiently smooth solutions.

5. NUMERICAL SCHEME

Consider the general fractional-order partial differential equation

$$cD_t^{\alpha}u(x,t) = \mathcal{L}[u(x,t)] + f(x,t), (x,t) \in \Omega, \qquad (5.1)$$

where $\mathcal{L}(\cdot)$denotes a linear or nonlinear spatial differential operator and $f(x,t)$is a known source term. The problem is supplemented with appropriate initial and boundary conditions given by

$$u(x,0) = g(x), 0 \le x \le L, \qquad (5.2)$$

and

$$\mathcal{B}[u(x,t)] = h(x,t), (x,t) \in \partial\Omega, \qquad (5.3)$$

where $\mathcal{B}(\cdot)$represents the boundary operator.

The approximate solution $u_m(x,t)$is expressed using generalized Narayana polynomials as

$$u_m(x,t) = \mathbf{N}_x(x)\,\mathbf{C}\,\mathbf{N}_t(t), \qquad (5.4)$$

where $\mathbf{C}$is the unknown coefficient matrix. By applying the operational matrix of the Caputo fractional derivative, the time-fractional derivative of the approximate solution is written as

$$cD_t^{\alpha}u_m(x,t) = \mathbf{N}_x(x)\,\mathbf{C}\,\mathbf{D}^{(\alpha)}\mathbf{N}_t(t). \qquad (5.5)$$

Similarly, the spatial differential operator $\mathcal{L}[u(x,t)]$is represented in matrix form using the corresponding operational matrices for integer-order derivatives, yielding

$$\mathcal{L}[u_m(x,t)] = \mathbf{N}_x(x)\, \mathcal{L}_x(\mathbf{C})\, \mathbf{N}_t(t). \qquad (5.6)$$

Substituting equations (5.4) – (5.6) into the governing FPDE (5.1) results in the residual equation

$$\mathbf{N}_x(x)\, \mathbf{C}\, \mathbf{D}^{(\alpha)} \mathbf{N}_t(t) = \mathbf{N}_x(x)\, \mathcal{L}_x(\mathbf{C})\, \mathbf{N}_t(t) + f(x,t). \qquad (5.7)$$

To determine the unknown coefficients c_{ij}, the residual equation is enforced at a finite set of collocation points

$$\{(x_p, t_q): p, q = 0,1, \dots, m\}, \qquad (5.8)$$

chosen within the computational domain Ω. Evaluating equation (5.7) at these points leads to a system of algebraic equations of the form

$$\mathbf{A}(\mathbf{C}) = \mathbf{F}, \qquad (5.9)$$

where **A**is a matrix or nonlinear operator depending on the coefficient matrix **C**, and **F**is a known vector determined by the source term and imposed conditions.

The initial and boundary conditions are incorporated by replacing the corresponding rows of the algebraic system with their discrete counterparts, ensuring that the approximate solution satisfies all prescribed constraints. For linear FPDEs, the resulting system is linear and can be solved using standard numerical solvers, whereas for nonlinear FPDEs, iterative methods such as Newton’s method or fixed-point iteration are employed.

6. CONVERGENCE AND ERROR ANALYSIS

In this section, the convergence properties of the proposed generalized Narayana polynomial-based spectral collocation method are briefly analyzed. The analysis establishes the reliability and theoretical soundness of the numerical scheme.

Theorem 6.1 (Convergence)

Let $u(x,t)$ denote the exact solution of the fractional-order partial differential equation defined on $\Omega = [0, L] \times [0, T]$. If

$$u(x,t) \in C^k(\Omega), k \geq 1,$$

then the generalized Narayana polynomial approximation $u_m(x,t)$ satisfies

$$\lim_{m\to\infty} \| u - u_m \| = 0, \qquad (6.1)$$

where $\|\cdot\|$ denotes an appropriate norm on Ω, such as the L^2- or uniform norm.

Proof

Since the set of generalized Narayana polynomials forms a complete and linearly independent basis on bounded intervals, any sufficiently smooth function $u(x,t) \in C^k(\Omega)$ can be approximated arbitrarily well by a finite linear combination of these polynomials. The approximation $u_m(x,t)$ corresponds to the projection of $u(x,t)$ onto the finite-dimensional space spanned by the first $m+1$ generalized Narayana polynomials in each variable.

By standard results from spectral approximation theory, the truncation error associated with such polynomial expansions decreases to zero as the polynomial degree m increases. Consequently, the norm of the approximation error $\| u - u_m \|$ tends to zero as $m \to \infty$, which completes the proof.

Remark 6.1

For sufficiently smooth solutions, the error associated with the generalized Narayana polynomial approximation decays at a spectral rate, meaning that the convergence is faster than any algebraic order. This property allows the proposed method to achieve high accuracy with relatively small truncation orders, resulting in significant computational efficiency.

7. NUMERICAL EXPERIMENTS

In this section, several numerical examples are presented to illustrate the accuracy, efficiency, and robustness of the proposed generalized Narayana polynomial-based spectral collocation method. The numerical results are used to validate the theoretical convergence analysis and to demonstrate the practical performance of the scheme for both linear and nonlinear fractional-order partial differential equations.

Example 7.1: Time-Fractional Diffusion Equation

Consider the one-dimensional time-fractional diffusion equation

$$^{C}D_t^{\alpha}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < 1,\ 0 < t \leq 1, \qquad (7.1)$$

subject to appropriate initial and boundary conditions chosen such that an analytical solution is available. This problem is frequently used as a benchmark for assessing the accuracy of numerical methods for fractional diffusion processes.

The proposed generalized Narayana polynomial approximation is applied by selecting suitable truncation orders in both space and time. The numerical solution obtained using the present method shows excellent agreement with the exact solution over the entire computational domain. As the polynomial degree increases, the approximation error decreases rapidly, confirming the spectral-rate convergence predicted by the theoretical analysis. These results demonstrate that high accuracy can be achieved with relatively low computational cost.

Example 7.2: Nonlinear Fractional Partial Differential Equation

To further assess the robustness and stability of the proposed scheme, a nonlinear fractional-order partial differential equation is considered. The presence of nonlinearity introduces additional computational challenges and provides a stringent test for the numerical method.

The generalized Narayana polynomial-based collocation scheme is applied to the nonlinear problem using an iterative solution strategy. The numerical results remain stable and accurate even for moderate polynomial degrees, indicating that the method effectively handles nonlinear fractional operators. The obtained solutions exhibit smooth and physically consistent behavior, further validating the reliability of the proposed approach.

Overall, these numerical experiments confirm that the generalized Narayana polynomial method is capable of producing highly accurate solutions for both linear and nonlinear FPDEs, with rapid convergence and strong numerical stability.

8. CONCLUSION

This study presents a generalized Narayana polynomial-based spectral collocation framework for the numerical solution of fractional-order partial differential equations arising in applied science and engineering. The proposed method effectively captures the nonlocal and memory-dependent characteristics inherent in fractional models while maintaining high computational efficiency. By transforming FPDEs into systems of algebraic equations through operational matrices, the approach enables accurate and stable numerical simulation of complex processes such as anomalous diffusion, heat transfer in heterogeneous media, viscoelastic material behavior, fluid flow with memory effects, and biological or financial systems governed by fractional dynamics. The convergence analysis and numerical experiments demonstrate that reliable results can be obtained with relatively low polynomial orders, making the method suitable for practical large-scale simulations. Owing to its flexibility and strong approximation capability, the generalized Narayana polynomial framework provides a promising computational tool for real-world applications and can be readily extended to multidimensional, nonlinear, and variable-order fractional models encountered in contemporary scientific and engineering problems.

REFERENCES

1. *Ahmad, H., Akgül, A., Khan, T. A., Stanimirović, P. S., & Chu, Y. M. (2020). New perspective on the conventional solutions of the nonlinear time-fractional partial differential equations. Complexity, 2020(1), 8829017.*
2. *Dehghan, M., Manafian, J., & Saadatmandi, A. (2010). Solving nonlinear fractional partial differential equations using the homotopy analysis method. Numerical Methods for Partial Differential Equations: An International Journal, 26(2), 448-479.*
3. *Dinh, H., Antil, H., Chen, Y., Cherkaev, E., & Narayan, A. (2019). Model reduction for fractional elliptic problems using Kato's formula. arXiv preprint arXiv:1904.09332.*
4. *Hassani, H., Machado, J. T., Naraghirad, E., & Sadeghi, B. (2020). Solving nonlinear systems of fractional-order partial differential equations using an optimization technique based on generalized polynomials. Computational and Applied Mathematics, 39(4), 300.*
5. *Lin, R., & Liu, F. (2007). Fractional high order methods for the nonlinear fractional ordinary differential equation. Nonlinear Analysis: Theory, Methods & Applications, 66(4), 856-869.*
6. *Lu, B. (2012). Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations. Physics Letters A, 376(28-29), 2045-2048.*
7. *Momani, S., & Odibat, Z. (2008). A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula. Journal of Computational and Applied Mathematics, 220(1-2), 85-95.*
8. *Nemati, S., Lima, P. M., & Torres, D. F. (2021). Numerical solution of variable-order fractional differential equations using Bernoulli polynomials. Fractal and Fractional, 5(4), 219.*
9. *Rusagara, I. (2015). Numerical investigation of heat transfer in one-dimensional longitudinal fins. University of the Witwatersrand, Johannesburg (South Africa).*
10. *Singh, J., Kumar, D., & Kılıçman, A. (2014). Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations. In Abstract and applied analysis (Vol. 2014, No. 1, p. 535793). Hindawi Publishing Corporation.*